Applied General Topology

Appl. Gen. Topol. 26, no. 1 (2025), 341-368
doi:10.4995/agt.2025.22003

$C_1(X)$ on the edge of $C_2(X)$

Javier Camargo [a], Sergio Macías [b] and David Maya [c]

[a] Escuela de Matemáticas, Facultad de Ciencias, Universidad Industrial de Santander, Ciudad Universitaria, Carrera 27 Calle 9, Bucaramanga, Santander, A. A. 678, Colombia. (jcamargo@saber.uis.edu.co)

[b] Current address: Escuela de Matemáticas, Facultad de Ciencias, Universidad Industrial de Santander, Ciudad Universitaria, Carrera 27 Calle 9, Bucaramanga, Santander, A. A. 678, Colombia
Permanent address: Instituto de Matemáticas, Universidad Nacional Autónoma de México, Circuito Exterior, Ciudad Universitaria, CDMX, C. P. 04510. México (macias@unam.mx, sergiom@matem.unam.mx)

[c] Universidad Autónoma del Estado de México, Facultad de Ciencias, Instituto Literario No. 100, Col. Centro, Toluca, Estado de México, CP 5000, México (dmayae@uaemex.mx)

Communicated by D. N. Georgiou

Abstract

Given a continuum X and a positive integer n, let $C_n(X)$ be the hyperspace consisting of all nonempty closed subsets of X having at most n components. For a subcontinuum A of X having empty interior, consider the following properties: A is a subcontinuum of colocal connectedness, $X \setminus A$ is continuumwise connected, A is a nonblock subcontinuum, A is a shore subcontinuum, A is not a strong centre. In this paper, we prove that $C_1(X)$ has all of these properties in $C_n(X)$ if $n \geq 3$, and we study when $C_1(X)$ has one of these properties in $C_2(X)$.

2020 MSC: *54B20; 54C60; 54F15.*

Keywords: *composant; continuum; continuum of colocal connectedness; continuumwise connected space; hyperspace; n-fold hyperspace; property of Kelley; property of Kelley weakly; not a strong centre; pseudo-arc; set function $\mathcal{T}$; shore subcontinuum; strongly continuumwise connected space; union composant; $\mathcal{T}$-closed set; $\mathcal{T}$-closed subcontinuum.*

Received 29 June 2024 – Accepted 12 December 2024

1. Introduction

For a continuum X, let $C_n(X)$ be the hyperspace of all nonempty closed subset of X having at most n-components endowed with the Hausdorff metric. The hyperspace $C_n(X)$ is a continuum [16, Corollary 1.8.12] and $C_1(X)$ is a subcontinuum of $C_n(X)$ having empty interior [16, Theorem 6.1.9]. By [5, Lemma 3.2], $C_n(X) \setminus C_1(X)$ is connected for every continuum X. The aim of this paper is to study when $C_1(X)$ is a subcontinuum of colocal connectedness, a nonblock subcontinuum, a shore subcontinuum or is not a strong centre of $C_n(X)$ for a continuum X.

The paper is divided in five sections, after this introduction 1 and the section of definitions 2, we have section 3. In this section we give conditions for which $C_1(X)$ is a subcontinuum of colocal connectedness of $C_n(X)$ (n an integer greater than or equal to 2). For example, we show that for $n \geq 3$, $C_1(X)$ is a subcontinuum of colocal connectedness of $C_n(X)$ (Theorem 3.1). We also prove that the fact that $C_1(X)$ is a subcontinuum of colocal connectedness of $C_2(X)$ implies that $F_1(X)$ is a subcontinuum of colocal connectedness of $F_2(X)$ (Theorem 3.2). We show that for aposyndetic continua, we have that $C_1(X)$ is a subcontinuum of colocal connectedness of $C_2(X)$ (Corollary 3.7). In section 4, we consider several classes of continua for which we have that $C_2(X) \setminus C_1(X)$ is continuumwise connected. In Theorem 4.1, we prove that $C_2(X) \setminus C_1(X)$ is continuumwise connected if and only if $F_2(X) \setminus F_1(X)$ is continuumwise connected. We also demonstrate that for Wilder continua $C_2(X) \setminus C_1(X)$ is continuumwise connected (Corollary 4.4); the same is true for λ-dendroid without terminal subcontinua (Theorem 4.13) and for compactifications of a ray with an arcwise connected continuum as remainder such that the family of subcontinua of the ray is not dense in the hyperspace of subcontinua of the continuum (Theorem 4.30). We prove that for an indecomposable chainable continuum X we have that $C_2(X) \setminus C_1(X)$ is not continuumwise connected (Corollary 4.23). In section 5, we give sufficient conditions on a continuum X such that $C_1(X)$ is a nonblock subcontinuum of $C_2(X)$. This is the case for the following classes of continua: continua with the property of Kelley weakly (Theorem 5.4); indecomposable chainable continua (Theorem 5.9); indecomposable continua such that each composant of it is a union composant (Theorem 5.13). We show that for decomposable continua, we have that the facts: $C_1(X)$ is a nonblock subcontinuum of $C_2(X)$, $C_1(X)$ is a shore subcontinuum of $C_2(X)$ and $C_1(X)$ is not a strong centre of $C_2(X)$ are equivalent (Theorem 5.15).

2. Definitions and auxiliary results

Given a metric space X with metric d, if $a \in X$ and $\varepsilon > 0$, then the *open ball* about a of radius ε is denoted by $B_d(a;\varepsilon)$. If A is a subset of X, then the interior of A is denoted by $\mathrm{Int}_X(A)$, its closure by $\mathrm{Cl}_X(A)$ and its diameter by $\mathrm{diam}(A)$. Also, we denote

$$\mathcal{N}_d(\varepsilon; A) = \{x \in X \mid d(x,a) < \varepsilon \text{ for some } a \in A\}.$$

A *continuum* is a compact connected metric space. In continuum theory, there are different concepts related with the edge of a continuum or being of some sort of non-cut subset of a continuum. A *subcontinuum* is a nonempty connected closed subset of a continuum.

Let X be a continuum and let A and B be two nonempty disjoint closed subsets of X. A subcontinuum K of X is *irreducible between* A *and* B provided that $K \cap A \neq \varnothing$, $K \cap B \neq \varnothing$ and no proper subcontinuum L of K satisfies both inequalities.

A continuum X is *decomposable* if there exist two proper subcontinua K and L of X such that $X = K \cup L$. Also, X is *hereditarily decomposable* provided that each of its nondegenerate subcontinua is decomposable. The continuum X is *indecomposable* provided that it is not decomposable. In addition, X is *hereditarily unicoherent* if the intersection of any two of its subcontinua is connected. A continuum X is *semi-aposyndetic* if for every pair of points p and q of X, there exists a subcontinuum W of X such that $\{p,q\} \cap \mathrm{Int}_X(W) \neq \varnothing$ and $\{p,q\} \cap (X \setminus W) \neq \varnothing$. The continuum X is *aposyndetic at a point* p *with respect to* q, provided that there exists a subcontinuum W of X such that $p \in \mathrm{Int}_X(W) \subseteq W \subseteq X \setminus \{q\}$. Now, X is *aposyndetic at* p if it is aposyndetic at p with respect to every point $q \in X \setminus \{p\}$. And X is *aposyndetic* provided that X is aposyndetic at each of its points.

Given a continuum X, we define Professor Jones' *Set Function* $\mathcal{T}$ as follows: if A is a subset of X, then

$$\mathcal{T}(A) = X \setminus \{x \in X \mid \text{there exists a subcontinuum } W \text{ of } X \text{ such that } x \in \mathrm{Int}_X(W) \subseteq W \subset X \setminus A\}.$$

Let us observe that for each subset A of X, $\mathcal{T}(A)$ is a closed subset of X and $A \subseteq \mathcal{T}(A)$. Note that, if W is a subcontinuum of X, then $\mathcal{T}(W)$ is a subcontinuum of X [18, Theorem 2.1.27]. Also, by [18, Corollary 2.1.14], $\mathcal{T}(\varnothing) = \varnothing$. A nonempty closed subset A of X is a $\mathcal{T}$*-closed set* if $\mathcal{T}(A) = A$. When there is a possibility of confusion, we add a subscript to $\mathcal{T}$ to indicate the continuum we are considering. More information about the set function $\mathcal{T}$ may be found in [18].

Let X be a continuum. We define 2^X the hyperspace of all nonempty closed subsets of X; i.e., $2^X = \{A \subseteq X \mid A \neq \varnothing, A = \mathrm{Cl}_X(A)\}$. We topologized this hyperspace with the topology induced by the *Hausdorff metric*, $\mathcal{H}$, given by

$$\mathcal{H}(A, B) = \inf\{\varepsilon > 0 \mid A \subseteq \mathcal{N}_d(\varepsilon; B) \text{ and } B \subseteq \mathcal{N}_d(\varepsilon; A)\},$$

for each $A, B \in 2^X$ [16, Theorem 1.8.3]. If $K_1, \dots, K_m$ are nonempty subsets of X, let

$$\langle K_1, \dots, K_m\rangle = \left\{A \in 2^X \;\middle|\; A \subset \bigcup_{j=1}^{m} K_j \text{ and } A \cap K_j \neq \varnothing \text{ for all } j \in \{1, \dots, m\}\right\}.$$

The topology induced by the Hausdorff metric coincides with the Vietoris topology on 2^X [22, Theorem (0.13)]. A base for this topology on 2^X is given by the sets of the form $\langle U_1, \dots, U_m\rangle$, where $U_1, \dots, U_m$ are open subsets of X. For each positive integer n, let

- $C_n(X) = \{A \in 2^X \mid A \text{ has at most } n \text{ components}\}$.
- $F_n(X) = \{A \in 2^X \mid |A| \leq n\}$.

Note that $F_n(X) \subseteq C_n(X) \subseteq 2^X$, for every $n \in \mathbb{N}$. Hence, $F_n(X)$ and $C_n(X)$ are topological subspaces of 2^X.

Let X be a continuum and let A be a subcontinuum of X such that $\mathrm{Int}_X(A) = \varnothing$. Then

- A is a *subcontinuum of colocal connectedness* of X if for each open subset U of X such that $A \subseteq U$, there exists an open subset V of X such that $A \subseteq V \subseteq U$ and $X \setminus V$ is connected.
- $X \setminus A$ is *continuumwise connected* provided that for any $x, y \in X \setminus A$ there exists a subcontinuum L of $X \setminus A$ such that $x, y \in L$.
- A is a *nonblock subcontinuum* of X if there exits an increasing sequence of subcontinua $\{M_n\}_{n=1}^{\infty}$ of X such that $\bigcup_{n=1}^{\infty} M_n$ is a dense subset of $X \setminus A$.
- A is a *shore subcontinuum* of X provided that for each $\varepsilon > 0$, there exists a subcontinuum L of $X \setminus A$ such that $\mathcal{H}(L, X) < \varepsilon$.
- A is not a *strong centre* of X if for any nonempty open subsets U and V of X there exists a subcontinuum M of X such that $M \cap U \neq \varnothing$, $M \cap V \neq \varnothing$ and $M \cap A = \varnothing$.
- A is a *noncut subcontinuum* of X provided that $X \setminus A$ is connected.

These notions are studied in the context of hyperspace of continua in [1, 9, 19].

Notation 2.1. If Z is a metric space (not necessarily a continuum), we denote by $C_1(Z)$ the collection of all subcontinua of Z.

Notation 2.2. Given a continuum X, a nonempty subset A of X and a point $p \in X \setminus A$, we denote

$$\kappa_{X\setminus A}(p) = \bigcup \{C \in C_1(X \setminus A) \mid p \in C\}.$$

Lemma 2.3. *Let X be a continuum and let A be a subcontinuum of X. Then A is a nonblock subcontinuum of X if and only if $\kappa_{X\setminus A}(p)$ (Notation 2.2) is a dense subset of X, for some $p \in X \setminus A$.*

Proof. Suppose that A is a nonblock subcontinuum of X. Then there exists an increasing sequence of subcontinua $\{M_n\}_{n=1}^{\infty}$ of $X \setminus A$ such that $\bigcup_{n=1}^{\infty} M_n$ is a dense subset of $X \setminus A$. Hence, let $p \in M_1$. Note that $\bigcup_{n=1}^{\infty} M_n \subseteq \kappa_{X\setminus A}(p)$. Thus, $\kappa_{X\setminus A}(p)$ is a dense subset of X.

Conversely, suppose that there exists $p \in X \setminus A$ such that $\kappa_{X\setminus A}(p)$ is dense. Let $\{U_n\}_{n=1}^{\infty}$ be a countable base of X. Given $n \in \mathbb{N}$, let L_n be a subcontinuum of X such that $p \in L_n \subseteq X \setminus A$ and $L_n \cap U_n \neq \varnothing$. If $M_n = L_1 \cup \cdots \cup L_n$, then $\{M_n\}_{n=1}^{\infty}$ is an increasing sequence of X such that $\bigcup_{n=1}^{\infty} M_n$ is a dense subset of $X \setminus A$; i.e., A is a nonblock subcontinuum of X. □

Next lemma follows from the definitions.

Lemma 2.4. *Let X be a continuum and let A be a subcontinuum of X. Consider the following statements:*

(1) *A is a subcontinuum of colocal connectedness of X;*

(2) *$X \setminus A$ is continuumwise connected;*

(3) *A is a nonblock subcontinuum of X;*

(4) *A is a shore subcontinuum of X;*

(5) *A is not a strong centre in X;*

(6) *A is a noncut subcontinuum in X.*

Then item (j) implies item (k), whenever $j \in \{1,2,3,4,5\}$, $k \in \{2,3,4,5,6\}$, and $j < k$.

In [3], the authors gave examples where it is shown that the converse implications in Lemma 2.4 are not true, for degenerate subcontinua.

Theorem 2.5. *Let X be a continuum and let A be a $\mathcal{T}$-closed subcontinuum of X. If $X \setminus A$ is connected, then A is a subcontinuum of colocal connectedness of X.*

Proof. Let $Y = X/A$ be the quotient space and let $q\colon X \to Y$ be the quotient map. Since $q|_{X\setminus A}\colon X \setminus A \to Y \setminus \{q(A)\}$ is a homeomorphism, we have that Y is semi-locally connected at

$q(A)$ [18, Theorem 2.1.35]. Observe that $Y\setminus\{q(A)\}$ is connected. Thus, Y is colocally connected at $\{q(A)\}$, by [24, (4.14), p. 50]. Therefore, A is a subcontinuum of colocal connectedness of X. □

Remark 2.6. Let X be a continuum and let A be a $\mathcal{T}$-closed subcontinuum of X. Observe that in this case, by Theorem 2.5, all the properties of Lemma 2.4 are equivalent. In fact, we can add that $X \setminus A$ is connected to [8, Theorem 4.15]. Also observe that it is not required that the subcontinuum A has empty interior in Theorem 2.5.

Remark 2.7. Note that, by [5, Lemma 3.2], we have that if X is a continuum and $n \in \mathbb{N}$, then $C_n(X)\setminus C_1(X)$ is always connected. Hence, $C_1(X)$ is always a noncut subcontinuum of $C_n(X)$.

A *map* is a continuous function. A map $f\colon X \to Y$ between continua is *open* provided that for each open subset U of X, $f(U)$ is an open subset of Y. A continuum which is the inverse limit of a sequence of arcs and open bonding maps is called *Knaster continuum*.

The next lemma is [19, Example 4.5].

Lemma 2.8. *If X is a Knaster continuum, then $F_1(X)$ is not a subcontinuum of colocal connectedness in $F_2(X)$.*

The next result follows immediately from [19, Theorem 4.8].

Corollary 2.9. *If $X = \{(x, \sin(\frac{1}{x})) \mid x \in (0,1]\} \cup (\{0\} \times [-1,1])$, then $F_2(X) \setminus F_1(X)$ is not continuumwise connected.*

An *order arc* is a map $\alpha\colon [0,1] \to \mathcal{R}$, where $\mathcal{R} \in \{2^X, C_n(X)\}$, such that if s and t belong to $[0,1]$ and $s < t$, then $\alpha(s) \subsetneq \alpha(t)$. When it is convenient, we identify α with its image $\alpha([0,1])$.

Notation 2.10. Given $A \in C_2(X)\setminus C_1(X)$ and $A = A_1 \cup A_2$, where A_1 and A_2 are the components of A, we denote $G_A = \langle A_1, A_2\rangle \cap F_2(X)$. By [20, Lemma 1], G_A is a subcontinuum of $F_2(X)$, for each $A \in C_2(X)\setminus C_1(X)$. Also, if $\mathcal{L}$ is a subcontinuum of $C_2(X)\setminus C_1(X)$, let $\mathcal{G}_{\mathcal{L}} = \bigcup\{G_A \mid A \in \mathcal{L}\}$.

Lemma 2.11. *Let X be a continuum. Let $\mathcal{G}\colon C_2(X) \setminus C_1(X) \to C_1(F_2(X))$ be given by*

$$\mathcal{G}(A) = G_A,$$

for each $A \in C_2(X) \setminus C_1(X)$ (Notation 2.10). Then $\mathcal{G}$ is a map.

Proof. Let $\{A_n\}_{n=1}^{\infty}$ be a sequence in $C_2(X) \setminus C_1(X)$ converging to A, for some $A \in C_2(X) \setminus C_1(X)$. We show that $\lim_{n\to\infty} \mathcal{G}(A_n) = \mathcal{G}(A)$. Let $A_n = C_n \cup D_n$, for each $n \in \mathbb{N}$ and let $A =$

$C \cup D$. Without loss of generality, we may assume that $\lim_{n\to\infty} C_n = C$ and $\lim_{n\to\infty} D_n = D$. We prove that $\mathcal{G}(A) \subseteq \liminf \mathcal{G}(A_n)$ and $\limsup \mathcal{G}(A_n) \subseteq \mathcal{G}(A)$.

- $\mathcal{G}(A) \subseteq \liminf \mathcal{G}(A_n)$.

Let $\{c, d\} \in \mathcal{G}(A)$, where $c \in C$ and $d \in D$. Then there exist two sequences $\{c_n\}_{n=1}^{\infty}$ and $\{d_n\}_{n=1}^{\infty}$ such that $c_n \in C_n$ and $d_n \in D_n$, for each $n \in \mathbb{N}$, $\lim_{n\to\infty} c_n = c$ and $\lim_{n\to\infty} d_n = d$. If $\mathcal{U}$ is an open subset of $F_2(X)$ containing $\{c, d\}$, then there exists $k \in \mathbb{N}$ such that $\{c_n, d_n\} \in \mathcal{U}$, for all $n \geq k$. Since $\{c_n, d_n\} \in \mathcal{G}(A_n)$, $\mathcal{G}(A_n) \cap \mathcal{U} \neq \varnothing$, for every $n \geq k$. Therefore, $\{c, d\} \in \liminf \mathcal{G}(A_n)$ and $\mathcal{G}(A) \subseteq \liminf \mathcal{G}(A_n)$.

- $\limsup \mathcal{G}(A_n) \subseteq \mathcal{G}(A)$.

Let $\{x, y\} \in \limsup \mathcal{G}(A_n)$. Then there exists a sequence $\{\{c_{n_k}, d_{n_k}\}\}_{k=1}^{\infty}$ such that $\{c_{n_k}, d_{n_k}\} \in \mathcal{G}(A_{n_k})$, for each $k \in \mathbb{N}$, and $\lim_{k\to\infty}\{c_{n_k}, d_{n_k}\} = \{x, y\}$. Since $\lim_{n\to\infty} C_n = C$ and $\lim_{n\to\infty} D_n = D$, $\{x, y\} \in \langle C, D\rangle \cap F_2(X)$. Thus, $\{x, y\} \in \mathcal{G}(A)$. Therefore, $\limsup \mathcal{G}(A_n) \subseteq \mathcal{G}(A)$.

Therefore, $\lim_{n\to\infty} \mathcal{G}(A_n) = \mathcal{G}(A)$ [22, Theorem (0.7)] and $\mathcal{G}$ is a map. □

Corollary 2.12. *Let X be a continuum. If $\mathcal{L}$ is a subcontinuum of $C_2(X) \setminus C_1(X)$, then $\mathcal{G}_{\mathcal{L}}$ (Notation 2.10) is a subcontinuum of $F_2(X) \setminus F_1(X)$ such that $\mathcal{L} \cap F_2(X) \subseteq \mathcal{G}_{\mathcal{L}}$.*

Proof. Note that $\mathcal{G}_{\mathcal{L}} = \bigcup \mathcal{G}(\mathcal{L}) = \bigcup\{G_A \mid A \in \mathcal{L}\}$. Hence, by [22, Lemma (1.48)] and Lemma 2.11, $\mathcal{G}_{\mathcal{L}}$ is a subcontinuum of $F_2(X)$. Since $\mathcal{G}(A) \cap F_1(X) = \varnothing$, for each $A \in C_2(X) \setminus C_1(X)$, we have that $\mathcal{G}_{\mathcal{L}}$ is a subcontinuum of $F_2(X) \setminus F_1(X)$.

Now, observe that if $A \in \mathcal{L} \cap F_2(X)$, then $A = \{x, y\}$, for some $x, y \in X$ and A is the unique element of the set $\langle\{x\}, \{y\}\rangle \cap F_2(X)$. Hence, $A \in \mathcal{G}_{\mathcal{L}}$ and $\mathcal{L} \cap F_2(X) \subseteq \mathcal{G}_{\mathcal{L}}$. □

3. Colocal connectivity of $C_1(X)$ in $C_n(X)$

In Remark 2.7, we showed that $C_n(X) \setminus C_1(X)$ is connected. We begin this section showing that for $n \geq 3$, we always have that $C_1(X)$ is a subcontinuum of colocal connectedness of $C_n(X)$.

Theorem 3.1. *Let X be a continuum and let $n \geq 3$. Then $C_1(X)$ is a subcontinuum of colocal connectedness of $C_n(X)$.*

Proof. Note that, by [5, Theorem 3.9], if $A_0 \in C_n(X) \setminus C_1(X)$, then there exists a subcontinuum $\mathcal{L}$ of $C_n(X) \setminus C_1(X)$ such that $A_0 \in \mathrm{Int}_{C_n(X)}(\mathcal{L})$; i.e., $C_1(X)$ is a $\mathcal{T}_{C_n(X)}$-closed set in $C_n(X)$. Thus, by Theorem 2.5, $C_1(X)$ is a subcontinuum of colocal connectedness of $C_n(X)$. □

Regarding the case $n = 2$, we have the following:

Theorem 3.2. *Let X be a continuum. If $C_1(X)$ is a continuum of colocal connectedness in $C_2(X)$, then $F_1(X)$ is a continuum of colocal connectedness in $F_2(X)$.*

Proof. Suppose that $C_1(X)$ is a subcontinuum of colocal connectedness of $C_2(X)$. Let $\mathcal{V}$ be an open subset of $F_2(X)$ such that $F_1(X) \subseteq \mathcal{V}$. Then there exists $\varepsilon > 0$ such that $\mathcal{N}_{\mathcal{H}|_{F_2(X)}}(\varepsilon; F_1(X)) \subseteq \mathcal{V}$. Note that $\mathcal{N}_{\mathcal{H}|_{C_2(X)}}(\varepsilon; C_1(X))$ is an open subset of $C_2(X)$ such that $C_1(X) \subseteq \mathcal{N}_{\mathcal{H}|_{C_2(X)}}(\varepsilon; C_1(X))$. Thus, there exists an open subset $\mathcal{W}$ of $C_2(X)$ such that

$$C_1(X) \subseteq \mathcal{W} \subseteq \mathcal{N}_{\mathcal{H}|_{C_2(X)}}(\varepsilon; C_1(X))$$

and $C_2(X) \setminus \mathcal{W}$ is connected. Observe that $C_2(X) \setminus \mathcal{W}$ is a continuum such that $(C_2(X) \setminus \mathcal{W}) \cap C_1(X) = \varnothing$. Let $\mathcal{L} = C_2(X) \setminus \mathcal{W}$. By Corollary 2.12, $\mathcal{G}_{\mathcal{L}}$ is a subcontinuum of $F_2(X) \setminus F_1(X)$ such that $\mathcal{L} \cap F_2(X) \subseteq \mathcal{G}_{\mathcal{L}}$. Let $\mathcal{U} = F_2(X) \setminus \mathcal{G}_{\mathcal{L}}$. Since $F_1(X) \cap \mathcal{G}_{\mathcal{L}} = \varnothing$, $F_1(X) \subseteq \mathcal{U}$.

To see that $\mathcal{U} \subseteq \mathcal{N}_{\mathcal{H}|_{F_2(X)}}(\varepsilon; F_1(X))$, let $\{x, y\} \in \mathcal{U}$. Since $\mathcal{U} = F_2(X) \setminus \mathcal{G}_{\mathcal{L}}$ and $F_2(X) \setminus \mathcal{G}_{\mathcal{L}} \subseteq F_2(X) \setminus (\mathcal{L} \cap F_2(X))$, we have that

$$\{x, y\} \in F_2(X) \setminus \mathcal{L} \subseteq C_2(X) \setminus \mathcal{L} = \mathcal{W} \subseteq \mathcal{N}_{\mathcal{H}|_{C_2(X)}}(\varepsilon; C_1(X)).$$

Thus, there exists $A \in C_1(X)$ such that $\mathcal{H}(\{x, y\}, A) < \varepsilon$. Note that $A \subseteq B_d(x; \varepsilon) \cup B_d(y; \varepsilon)$, $A \cap B_d(x; \varepsilon) \neq \varnothing$ and $A \cap B_d(y; \varepsilon) \neq \varnothing$. Since A is connected, $B_d(x; \varepsilon) \cap B_d(y; \varepsilon) \neq \varnothing$. Let $z \in B_d(x; \varepsilon) \cap B_d(y; \varepsilon)$. Thus, $\mathcal{H}(\{x, y\}, \{z\}) < \varepsilon$, and $\{x, y\} \in \mathcal{N}_{\mathcal{H}|_{F_2(X)}}(\varepsilon; F_1(X))$. Therefore, $\mathcal{U} \subseteq \mathcal{N}_{\mathcal{H}|_{F_2(X)}}(\varepsilon; F_1(X))$. Since $\mathcal{N}_{\mathcal{H}|_{F_2(X)}}(\varepsilon; F_1(X)) \subseteq \mathcal{V}$, we obtain that $\mathcal{U} \subseteq \mathcal{V}$. □

Question 3.3. *Let X be a continuum such that $F_1(X)$ is a subcontinuum of colocal connectedness of $F_2(X)$. Then does it follow that $C_1(X)$ is a subcontinuum of colocal connectedness of $C_2(X)$?*

The next result follows from Theorem 3.2 and Lemma 2.8 together.

Corollary 3.4. *If X is a Knaster continuum, then $C_1(X)$ is not a subcontinuum of colocal connectedness of $C_2(X)$.*

The continuum exhibited in the next example is an arcwise connected continuum X such that $C_1(X)$ is not a subcontinuum of colocal connectedness of $C_2(X)$.

Example 3.5. Let A be convex arc in $\mathbb{R}^2$ between $(-1, 0)$ and $(1, 0)$, and for each $n \in \mathbb{N}$, let L_n be the convex arc in $\mathbb{R}^2$ between $(1, 0)$ and $(0, \frac{1}{n})$ and let J_n be the convex arc in $\mathbb{R}^2$ between $(-1, 0)$ and $(0, -\frac{1}{n})$. Let $X = A \cup \bigcup_{n \in \mathbb{N}} (L_n \cup J_n)$. By [19, Example 4.7], $F_1(X)$ is not a subcontinuum of colocal connectedness of $F_2(X)$. This and Theorem 3.2 show that $C_1(X)$ is not a subcontinuum of colocal connectedness of $C_2(X)$.

Corollary 3.6. *Let X be a continuum. Then the following are equivalent:*

(1) *$C_1(X)$ is a subcontinuum of colocal connectedness of $C_2(X)$;*

(2) *$C_1(X)$ is a $\mathcal{T}_{C_2(X)}$-closed subcontinuum of $C_2(X)$.*

Proof. Suppose that $C_1(X)$ is a subcontinuum of colocal connectedness of $C_2(X)$ and let $A \in C_2(X) \setminus C_1(X)$. Since $C_1(X)$ is a subcontinuum of colocal connectedness of $C_2(X)$, there exists an open set $\mathcal{U}$ such that $C_1(X) \subseteq \mathcal{U}$, $A \in C_2(X) \setminus \mathrm{Cl}_{C_2(X)}(\mathcal{U})$ and $C_2(X) \setminus \mathcal{U}$ is a continuum. Hence, $\mathcal{L} = C_2(X) \setminus \mathcal{U}$ is a subcontinuum of $C_2(X) \setminus C_1(X)$ such that $A \in \mathrm{Int}_{C_2(X)}(\mathcal{L})$.

The converse implication follows from Remark 2.7 and Theorem 2.5. □

More can be said if the continuum is aposyndetic.

Corollary 3.7. *If X is an aposyndetic continuum, then we have:*

(1) *$C_1(X)$ is a subcontinuum of colocal connectedness of $C_2(X)$*

and

(2) *$F_1(X)$ is a subcontinuum of colocal connectedness of $F_2(X)$.*

Proof. By [5, Theorem 3.9], if $A_0 \in C_2(X) \setminus C_1(X)$, then there exists a subcontinuum $\mathcal{L}$ of $C_2(X) \setminus C_1(X)$ such that $A_0 \in \mathrm{Int}_{C_2(X)}(\mathcal{L})$; i.e., $C_1(X)$ is a $\mathcal{T}_{C_2(X)}$-closed subcontinuum of $C_2(X)$. Since, by Remark 2.7, $C_2(X) \setminus C_1(X)$ is connected, by Theorem 2.5, $C_1(X)$ is a subcontinuum of colocal connectedness of $C_2(X)$.

By Theorem 3.2 and the part (1) of this theorem, we have that (2) holds. □

Remark 3.8. Observe that if X is the harmonic fan [16, Example 1.7.5], X is not aposyndetic, and it is not difficult to prove that $F_1(X)$ is a subcontinuum of colocal connectedness of $F_2(X)$. Hence, the converse of Corollary 3.7 is not true.

Remark 3.9. Note that, by Corollary 3.7, Question 3.3 has a positive answer for aposyndetic continua.

A continuum X is *colocally connected* provided that $\{x\}$ is a subcontinuum of colocal connectedness of X for each $x \in X$. Note that each colocally connected continuum is aposyndetic [16, Remark 5.4.15]. Hence, from Corollary 3.7, we have the next result.

Corollary 3.10. *If X is a colocally connected continuum, then we have:*

(1) *$F_1(X)$ is a subcontinuum of colocal connectedness of $F_2(X)$*

and

(2) *$C_1(X)$ is a subcontinuum of colocal connectedness of $C_2(X)$.*

Remark 3.11. Every aposyndetic continuum having at least one cut point shows that the converse of Corollary 3.10 fails.

4. Continuumwise connectivity of $C_2(X) \setminus C_1(X)$

We consider several classes of continua for which we have that $C_2(X) \setminus C_1(X)$ is continuumwise connected.

Theorem 4.1. *Let X be a continuum. Then the following are equivalent:*

(1) *$F_2(X) \setminus F_1(X)$ is continuumwise connected;*

(2) *$C_2(X) \setminus C_1(X)$ is continuumwise connected.*

Proof. Suppose that $F_2(X) \setminus F_1(X)$ is continuumwise connected. We prove that $C_2(X) \setminus C_1(X)$ is continuumwise connected. Let $A, B \in C_2(X) \setminus C_1(X)$. Let $A = A_1 \cup A_2$ and $B = B_1 \cup B_2$, where A_1 and A_2 are the components of A, and B_1 and B_2 are the components of B. Let $a_i \in A_i$ and $b_i \in B_i$, for each $i \in \{1, 2\}$. By [22, Theorem (1.8)], there exist order arcs α and β in $C_2(X) \setminus C_1(X)$ from $\{a_1, a_2\}$ to A and from $\{b_1, b_2\}$ to B, respectively. Since $F_2(X) \setminus F_1(X)$ is continuumwise connected, there exists a subcontinuum $\mathcal{L}$ of $F_2(X) \setminus F_1(X)$ such that $\{a_1, a_2\}, \{b_1, b_2\} \in \mathcal{L}$. Observe that $\alpha \cup \beta \cup \mathcal{L}$ is a subcontinuum of $C_2(X) \setminus C_1(X)$ such that $A, B \in \alpha \cup \beta \cup \mathcal{L}$. Therefore, $C_2(X) \setminus C_1(X)$ is continuumwise connected.

Conversely, we suppose that $C_2(X) \setminus C_1(X)$ is continuumwise connected. Let $\{a_1, a_2\}, \{b_1, b_2\} \in F_2(X) \setminus F_1(X)$. Since $\{a_1, a_2\}$ and $\{b_1, b_2\}$ belong to $C_2(X) \setminus C_1(X)$, there exists a subcontinuum $\mathcal{L}$ of $C_2(X) \setminus C_1(X)$ such that $\{a_1, a_2\}$ and $\{b_1, b_2\}$ are in $\mathcal{L}$. By Corollary 2.12, $\mathcal{G}_{\mathcal{L}}$ is a subcontinuum of $F_2(X) \setminus F_1(X)$, and $\{a_1, a_2\}, \{b_1, b_2\} \in \mathcal{G}_{\mathcal{L}}$. Therefore, $F_2(X) \setminus F_1(X)$ is continuumwise connected. □

A continuum X is called *Wilder* provided that for any three points x, y and z of X, there exists a subcontinuum of X containing x and exactly one of y and z. Each arcwise connected continuum is Wilder [10, p. 2].

Lemma 4.2. *If X is a Wilder continuum and $x, y, z \in X$ are distinct, then there exists a subcontinuum $\mathcal{K}$ of $F_2(X)$ containing $\{x, y\}$ and $\{x, z\}$, and missing $F_1(X)$.*

Proof. Let T be a subcontinuum of X containing y and exactly one of x and z. If $z \in T$ and $x \notin T$, then $\langle \{x\}, T \rangle \cap F_2(X)$ is the required subcontinuum [20, Lemma 1]. Now, assume that $x \in T$ and $z \notin T$. Let Q be a subcontinuum of X such that $z \in Q$ and $Q \cap \{x, y\}$ is a singleton. If $x \notin Q$ and $y \in Q$, then the subcontinuum $\langle\{x\}, T\rangle \cap F_2(X)$ satisfies the required properties. Suppose that $x \in Q$ and $y \notin Q$. Let $\mathcal{K} = (\langle\{z\}, T\rangle \cup \langle\{y\}, Q\rangle) \cap F_2(X)$. Since $\{y, z\} \in \langle\{z\}, T\rangle \cap \langle\{y\}, Q\rangle$, $\mathcal{K}$ is a subcontinuum of $F_2(X)$. Observe that $\{x, y\}$ and $\{x, z\}$ both belong to $\mathcal{K}$. Since $y \notin Q$ and $z \notin T$, we have that $\mathcal{K} \cap F_1(X) = \varnothing$. Therefore, $\mathcal{K}$ satisfies the required properties. □

In [19, Theorem 4.6], it is mentioned that if X is an arcwise connected continuum, then $F_2(X) \setminus F_1(X)$ is also arcwise connected. Hence, $F_2(X) \setminus F_1(X)$ is continuumwise connected. In the following result, we generalize [19, Theorem 4.6] by showing that $F_2(X) \setminus F_1(X)$ is continuumwise connected when X is a Wilder continuum.

Theorem 4.3. *If X is a Wilder continuum, then $F_2(X) \setminus F_1(X)$ is continuumwise connected.*

Proof. Let $\{x, y\}$ and $\{w, z\}$ be two distinct elements of $F_2(X) \setminus F_1(X)$. If $\{x, y\} \cap \{w, z\} \neq \varnothing$, then there exists a subcontinuum $\mathcal{K}$ of $F_2(X)$ containing $\{x, y\}$ and $\{w, z\}$, and missing $F_1(X)$ (Lemma 4.2). Now, assume that $\{x, y\} \cap \{w, z\} = \varnothing$. By Lemma 4.2, there exist two subcontinua $\mathcal{K}_1$ and $\mathcal{K}_2$ of $F_2(X)$ such that $\{x, y\}, \{w, y\} \in \mathcal{K}_1 \subseteq F_2(X) \setminus F_1(X)$ and $\{w, y\}, \{w, z\} \in \mathcal{K}_2 \subseteq F_2(X) \setminus F_1(X)$. Then $\mathcal{K}_1 \cup \mathcal{K}_2$ is a subcontinuum of $F_2(X)$ containing $\{x, y\}$ and $\{w, z\}$, and contained in $F_2(X) \setminus F_1(X)$. Therefore, $F_2(X) \setminus F_1(X)$ is continuumwise connected. □

The next result follows from Theorems 4.1 and 4.3.

Corollary 4.4. *If X is a Wilder continuum, then $C_2(X) \setminus C_1(X)$ is continuumwise connected.*

Remark 4.5. In Example 3.5, it is shown an arcwise connected continuum such that $C_1(X)$ is not a subcontinuum of colocal connectedness of $C_2(X)$ and $F_1(X)$ is not a subcontinuum of colocal connectedness of $F_2(X)$. Thus, since every arcwise connected is Wilder [10, p. 2], Theorem 4.3 and Corollary 4.4 cannot be improved.

A continuum X is *freely decomposable* if for each pair of points p and q of X, there exist subcontinua P and Q of X such that $X = P \cup Q$, $p \in P \setminus Q$ and $q \in Q \setminus P$. Each freely decomposable continuum is Wilder (see [6, Theorem 3.10]).

Corollary 4.6. *If X is a freely decomposable continuum, then we have:*

(1) $F_2(X) \setminus F_1(X)$ *is continuumwise connected,*

and

(2) $C_2(X) \setminus C_1(X)$ *is continuumwise connected.*

By [10, Theorem 3.6], every semi-aposyndetic continuum is Wilder. Hence, by Theorem 4.1 and Corollary 4.4, we obtain:

Corollary 4.7. *Let X be a semi-aposyndetic continuum. Then we obtain:*

(1) $F_2(X) \setminus F_1(X)$ *is continuumwise connected,*

and

(2) $C_2(X) \setminus C_1(X)$ *is continuumwise connected.*

A continuum X is a *continuum-chainable continuum* provided that for each $\varepsilon > 0$ and every pair of points x_1 and x_2 of X, there exists an ε-chain of continua $\{K_1, \ldots, K_n\}$ (i.e., $K_j \cap K_l \neq \varnothing$ if and only if $|j - l| \leq 1$ and $\text{diam}(K_j) < \varepsilon$, for all j and l in $\{1, \ldots, n\}$) such that $x_1 \in K_1$ and $x_2 \in K_n$. Note that every continuum-chainable continuum is a Wilder continuum [10, Theorem 3.2]. Thus, by Theorem 4.1 and Corollary 4.4, we have:

Corollary 4.8. *Let X be a continuum-chainable continuum. Then*

(1) $F_2(X) \setminus F_1(X)$ *is continuumwise connected,*

and

(2) $C_2(X) \setminus C_1(X)$ *is continuumwise connected.*

Remark 4.9. Note that, with respect to Corollary 4.8, we can say more since, by [7, Theorem 3.12], we have that if X is a continuum-chainable continuum, then $C_2(X) \setminus C_1(X)$ is arcwise connected.

A continuum X is a *strongly Wilder continuum* provided that if x, y and z are three distinct points of X, there exists a subcontinuum K of X such that $\{x, y\} \subseteq K$ and $z \in X \setminus K$. Note that colocally connected continua are strongly Wilder [6, Proposition 3.9].

Corollary 4.10. *If X is a strongly Wilder continuum, then*

(1) $F_2(X) \setminus F_1(X)$ *is continuumwise connected,*

and

(2) $C_2(X) \setminus C_1(X)$ *is continuumwise connected.*

Proof. The condition on X implies that X is a Wilder continuum. By Theorem 4.3, the space $F_2(X) \setminus F_1(X)$ is continuumwise connected. Now, by Theorem 4.1, $C_2(X) \setminus C_1(X)$ is continuumwise connected. □

Example 4.15 shows that the converse of Corollary 4.10 is not true.

Lemma 4.11. *Let X be a continuum such that there exist proper subcontinua M and N of X, where $X = M \cup N$. Let $p \in M \setminus N$ and $q \in N \setminus M$. If A and B are proper disjoint subcontinua of X such that $A \setminus N \neq \varnothing$ and $B \cap N \neq \varnothing$, then there exists a subcontinuum $\mathcal{L}$ of $C_2(X) \setminus C_1(X)$ such that $\{A \cup B, \{p, q\}\} \subseteq \mathcal{L}$.*

Proof. Let $a \in A \setminus N$ and $b \in B \cap N$. Let α_1 and α_2 be order arcs from $\{b\}$ to N, and from $\{q\}$ to N, respectively [22, Theorem (1.8)]. Note that

$$\mathcal{A} = \{\{a\} \cup I \mid I \in \alpha_1 \cup \alpha_2\}$$

is a subcontinuum of $C_2(X) \setminus C_1(X)$ such that $\{\{a, b\}, \{a, q\}\} \subseteq \mathcal{A}$. Let β be an order arc from $\{a\}$ to M and let $\mathcal{B} = \{\{q\} \cup I \mid I \in \beta\}$. Thus, $\mathcal{B}$ is a subcontinuum of $C_2(X) \setminus C_1(X)$ and $\{\{a, q\}, \{p, q\}\} \subseteq \mathcal{B}$. Hence, $\mathcal{A} \cup \mathcal{B}$ is a subcontinuum of $C_2(X) \setminus C_1(X)$ such that $\{\{a, b\}, \{p, q\}\} \subseteq \mathcal{A} \cup \mathcal{B}$. Finally, there exists an order arc γ from $\{a, b\}$ to $A \cup B$. Therefore, $\mathcal{L} = \gamma \cup \mathcal{A} \cup \mathcal{B}$ is a subcontinuum of $C_2(X) \setminus C_1(X)$ such that $\{A \cup B, \{p, q\}\} \subseteq \mathcal{L}$. □

A subcontinuum K of a continuum X is *terminal* provided that if L is a subcontinuum of X satisfying that $K \cap L \neq \varnothing$, then either $K \subseteq L$ or $L \subseteq K$.

A λ-*dendroid* is a hereditarily decomposable and hereditarily unicoherent continuum.

Lemma 4.12. *Let X be a λ-dendroid. Let A_1 and A_2 be disjoint subcontinua of X and let L be an irreducible subcontinuum of X between A_1 and A_2. If L is not terminal, then there exists a subcontinuum S of X such that $S \setminus L \neq \varnothing$, and either $S \cap A_1 \neq \varnothing$ and $A_2 \setminus S \neq \varnothing$, or $S \cap A_2 \neq \varnothing$ and $A_1 \setminus S \neq \varnothing$.*

Proof. Let A_1 and A_2 be subcontinua of X such that $A_1 \cap A_2 = \varnothing$, and let L be an irreducible subcontinuum of X between A_1 and A_2. Since X is hereditarily decomposable, there exist two proper subcontinua M and N of L such that $L = M \cup N$. We assume that $A_1 \cap L \subseteq M \setminus N$ and $A_2 \cap L \subseteq N \setminus M$. Since L is not a terminal subcontinuum of X, there exists a subcontinuum R of X such that $R \cap L \neq \varnothing$, $R \setminus L \neq \varnothing$ and $L \setminus R \neq \varnothing$. Since X is hereditarily unicoherent, we have that $R \cap L$ is a proper subcontinuum of L. We consider two cases:

Case (1). $R \cap L \cap A_1 = \varnothing$.

If $R \cap L \cap N \neq \varnothing$, then $S = R \cup N$ is a subcontinuum of X such that $S \cap A_2 \neq \varnothing$, $A_1 \setminus S = \varnothing$ and $S \setminus L \neq \varnothing$. If $R \cap L \cap N = \varnothing$, then $S = R \cup M$ satisfies the conditions of the lemma.

Case (2). $R \cap L \cap A_1 \neq \varnothing$.

Observe that, in this situation, we have that $R \cap L \cap A_2 = \varnothing$, as in Case (1), we find S satisfying all conditions in lemma. □

Theorem 4.13. *Let X be a λ-dendroid. If X does not contain a nondegenerate proper terminal subcontinuum, then $C_2(X) \setminus C_1(X)$ is continuumwise connected.*

Proof. Let M and N be proper subcontinua of X such that $X = M \cup N$. Let $p \in M \setminus N$ and $q \in N \setminus M$. Let $D \in C_2(X) \setminus C_1(X)$, and let A and B be the components of D. We show that there exists a subcontinuum $\mathcal{L}$ of $C_2(X) \setminus C_1(X)$ such that $\{D, \{p, q\}\} \subseteq \mathcal{L}$. To this end, let L be an irreducible subcontinuum of X between A and B. We consider two cases.

Case (1). $L \setminus N \neq \varnothing$ and $L \setminus M \neq \varnothing$.

Note that $L \cap M$ and $L \cap N$ are proper subcontinua of L such that $L = (L \cap M) \cup (L \cap N)$. Since L is irreducible between A and B, we may assume that $A \cap L \subseteq L \setminus N$ and $B \cap L \subseteq L \setminus M$. Thus, $A \setminus N \neq \varnothing$ and $B \cap N \neq \varnothing$. By Lemma 4.11, there exists a subcontinuum $\mathcal{L}$ of $C_2(X) \setminus C_1(X)$ such that $\{D, \{p, q\}\} \subseteq \mathcal{L}$.

Case (2). $L \subseteq M$ or $L \subseteq N$.

Suppose that $L \subseteq M$. Let J be an irreducible subcontinuum of X between $\{q\}$ and L. We consider two subcases:

Subcase (2.1). $L \setminus J \neq \varnothing$.

Since $L \cap J$ is a continuum and L is irreducible between A and B, by Lemma 4.12, we may assume that $A \setminus J \neq \varnothing$ and $B \cap J \neq \varnothing$. Let $Z = M \cup J$. Note that M and J are proper subcontinua of Z. Let $p' \in M \setminus J$. Since $A \cap M \neq \varnothing$ and $B \cap J \neq \varnothing$, we have that there exists a subcontinuum $\mathcal{L}_1$ of $C_2(Z) \setminus C_1(Z)$ such that $\{D, \{p', q\}\} \subseteq \mathcal{L}_1$, as stated in Lemma 4.11. Furthermore, also by Lemma 4.11, there exists a subcontinuum $\mathcal{L}_2$ of $C_2(X) \setminus C_1(X)$ such that $\{\{p, q\}, \{p', q\}\} \subseteq \mathcal{L}_2$. Thus, we define $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2$, which is a subcontinuum of $C_2(X) \setminus C_1(X)$ and $\{\{p, q\}, D\} \subseteq \mathcal{L}$.

Subcase (2.2). $L \subseteq J$.

Since X does not have any terminal subcontinuum, by Lemma 4.12, there exists a continuum T such that $T \cap A \neq \varnothing$, $B \setminus T \neq \varnothing$ and $T \setminus L \neq \varnothing$. Let $b_0 \in B \setminus T$. Using similar ideas as in Subcase (2.1) and by Lemma 4.11, there exists a subcontinuum $\mathcal{L}$ of $C_2(M \cup J) \setminus C_1(M \cup J)$ such that $\{T \cup A \cup \{b_0\}, \{p, q\}\} \subseteq \mathcal{L}$.

Let α and β be order arcs from A to $T \cup A$, and from $\{b_0\}$ to B, respectively [22, Theorem (1.8)]. We define $\mathcal{E} = \{E \cup \{b_0\} \mid E \in \alpha\}$ and $\mathcal{F} = \{A \cup D \mid D \in \beta\}$. Observe that $\mathcal{E}$ and $\mathcal{F}$ are subcontinua of $C_2(X) \setminus C_1(X)$ such that $T \cup A \cup \{b_0\} \in \mathcal{E}$, $A \cup B \in \mathcal{F}$ and $A \cup \{b_0\} \in \mathcal{E} \cap \mathcal{F}$. Thus, $\mathcal{L} \cup \mathcal{E} \cup \mathcal{F}$ is a subcontinuum of $C_2(X) \setminus C_1(X)$ such that $\{D, \{p, q\}\} \subseteq \mathcal{L} \cup \mathcal{E} \cup \mathcal{F}$.

Therefore, $C_2(X) \setminus C_1(X)$ is continuumwise connected. □

Next result follows from Theorems 4.1 and 4.13.

Corollary 4.14. *Let X be a λ-dendroid. If X does not contain a nondegenerate proper terminal subcontinuum, then $F_2(X) \setminus F_1(X)$ is continuumwise connected.*

The continuum in the next example shows that the converse of Corollary 4.10 is not true.

Example 4.15. There exists a λ-dendroid X such that X is not strongly Wilder and $F_2(X) \setminus F_1(X)$ is continuumwise connected.

Let X be the plane continuum $(\{0\} \times [-1, 2]) \cup \{(x, \sin(\frac{1}{x})) \mid x \in (0, 1]\}$. Note that X is a λ-dendroid and X does not contain a nondegenerate proper terminal subcontinuum. By Corollary 4.14, $F_2(X) \setminus F_1(X)$ is continuumwise connected. Finally, observe that each subcontinuum K of X containing $(0, 0)$ and $(0, -1)$ contains $(0, 1)$. Hence, X is not strongly Wilder.

Remark 4.16. Let X be the $\sin(\frac{1}{x})$-continuum. Then X is a λ-dendroid containing a nondegenerate proper terminal subcontinuum. By Corollary 2.9, $F_2(X) \setminus F_1(X)$ is not continuumwise connected. In conclusion, the conditions on X of Corollary 4.14 can not be omitted.

Question 4.17. *Let X be a hereditarily decomposable continuum. If X does not contain a nondegenerate proper terminal subcontinuum, then does it follow that $F_2(X) \setminus F_1(X)$ is continuumwise connected?*

Remark 4.18. Note that, by Theorem 4.19 and Corollary 4.7, the answer to Question 4.17 is affirmative for the class of semi-aposyndetic continua. A similar proof to the one given in [18, Lemma 1.4.52] shows that semi-aposyndetic continua do not contain nondegenerate proper terminal subcontinua.

Let X be a continuum. The *diagonal* of X^2 is the set $\Delta_X = \{(x, x) \mid x \in X\}$. By [15, Lemma 11], we have that if X is an arc-like continuum, then $X^2 \setminus \Delta_X$ is not continuumwise connected. In Theorem 4.22, we prove that $F_2(X) \setminus F_1(X)$ is not continuumwise connected, when X is an indecomposable arc-like continuum.

Theorem 4.19. *Let X be an indecomposable chainable continuum. Then:*

(1) *If $\mathcal{L}$ is a subcontinuum of $F_2(X) \setminus F_1(X)$, then $\bigcup \mathcal{L} \neq X$.*

(2) *If $\mathcal{L}$ is a subcontinuum of $C_2(X) \setminus C_1(X)$, then $\bigcup \mathcal{L} \neq X$.*

Proof. We prove (1). Suppose that $\mathcal{L}$ is a subcontinuum of $F_2(X) \setminus F_1(X)$ such that $\bigcup \mathcal{L} = X$. Let $f_2 \colon X^2 \to F_2(X)$ be the map given by $f_2((x, y)) = \{x, y\}$, for each $(x, y) \in X^2$. It is well known that f_2 is open [15, Lemma 9]. Hence, there exists a continuum $\mathcal{A}$ in X^2 such that $f_2(\mathcal{A}) = \mathcal{L}$ [24, Theorem (7.5), p. 148]. Note that $\mathcal{A} \cap \Delta_X = \varnothing$. We denote the natural projections from X^2 onto X by π_1 and π_2, respectively.

Claim. $\pi_1(\mathcal{A}) = X$ or $\pi_2(\mathcal{A}) = X$.

Since $\bigcup \mathcal{L} = X$, there exist three points $\{a_1, a_2\}, \{b_1, b_2\}$ and $\{c_1, c_2\}$ in $\mathcal{L}$ such that a_1, b_1 and c_1 belong to different composants of X. Thus, there exist $a, b, c \in \mathcal{A}$ such that $f_2(a) = \{a_1, a_2\}$, $f_2(b) = \{b_1, b_2\}$ and $f_2(c) = \{c_1, c_2\}$. Hence, we have that either $|\{\pi_1(a), \pi_1(b), \pi_1(c)\} \cap \{a_1, b_1, c_1\}| \geq 2$ or $|\{\pi_2(a), \pi_2(b), \pi_2(c)\} \cap \{a_1, b_1, c_1\}| \geq 2$. In any case, since $\mathcal{A}$ is a continuum and X is irreducible between any two points in $\{a_1, b_1, c_1\}$, we have that $\pi_1(\mathcal{A}) = X$ or $\pi_2(\mathcal{A}) = X$.

Suppose that $\pi_1(\mathcal{A}) = X$. Since $\mathcal{A} \cap \Delta_X = \varnothing$, there exists $\varepsilon > 0$ such that $d(x, y) > \varepsilon$, for all $(x, y) \in \mathcal{A}$. Also, since X is arc-like, there exists an ε-map $f \colon X \to [0, 1]$. Note that $f \times f \colon X \times X \to [0, 1] \times [0, 1]$ is an ε-map (we use the "max" metric on $X \times X$). Hence, $(f \times f)(\mathcal{A}) \cap (f \times f)(\Delta_X) = \varnothing$, otherwise, $f \times f$ would not be an ε-map. Let $p_1 \colon [0, 1] \times [0, 1] \to [0, 1]$ and $p_2 \colon [0, 1] \times [0, 1] \to [0, 1]$ be given by $p_1((s, t)) = s$ and $p_2((s, t)) = t$; i.e, these are the projection maps from the square onto the unit interval. Since $p_1((f \times f)(\mathcal{A})) = f(\pi_1(\mathcal{A})) = [0, 1]$ and $p_2((f \times f)(\Delta_X)) = f(\pi_2(\Delta_X)) = [0, 1]$, we have that $(f \times f)(\mathcal{A}) \cap (f \times f)(\Delta_X) \neq \varnothing$ [21, Theorem 130, p. 158], a contradiction. Therefore, $\bigcup \mathcal{L} \neq X$. A similar contradiction is obtained if $\pi_2(\mathcal{A}) = X$.

To show (2). Note that if $\mathcal{L}$ is a subcontinuum of $C_2(X) \setminus C_1(X)$, then $\mathcal{G}_{\mathcal{L}}$ is a subcontinuum of $F_2(X) \setminus F_1(X)$ (Corollary 2.12), where $\bigcup \mathcal{L} = \bigcup \mathcal{G}_{\mathcal{L}}$. Therefore, (2) follows from (1). □

Remark 4.20. Let X be a decomposable continuum, let A and B be proper subcontinua of X such that $X = A \cup B$, and let $a \in A \setminus B$ and $b \in B \setminus A$. If $\mathcal{L} = \{\{a, z\} \mid z \in B\} \cup \{\{w, b\} \mid w \in A\}$, then $\mathcal{L}$ is a subcontinuum of $F_2(X) \setminus F_1(X)$ such that $\bigcup \mathcal{L} = X$. Thus, Theorem 4.19 does not hold if X is not an indecomposable continuum.

Remark 4.21. Let X be the indecomposable continuum defined in [13, Theorem 3.2]. Note that $F_1(X)$ is a subcontinuum of colocal connectedness of $F_2(X)$ [14, Corollary 8]. If a, b, c are distinct points in different composants of X, then there exists a continuum $\mathcal{L} \subseteq F_2(X) \setminus F_1(X)$ such that $\{a, b\}, \{a, c\} \in \mathcal{L}$. Since $\bigcup \mathcal{L} \in C_2(X)$ and $\{a, b, c\} \subseteq \bigcup \mathcal{L}$, we have that $\bigcup \mathcal{L} = X$. Thus, Theorem 4.19 does not hold if X is not chainable.

Theorem 4.22. *Let X be an indecomposable chainable continuum. Then $F_2(X) \setminus F_1(X)$ is not continuumwise connected.*

Proof. Let a, b, c, d be distinct points in different composants of X (the composants of indecomposable continua are pairwise disjoint [12, Theorem 3-47]). If $\mathcal{L}$ is a subcontinuum of $F_2(X)$ such that $\{\{a, b\}, \{c, d\}\} \subseteq \mathcal{L}$, then $\bigcup \mathcal{L} \in C_2(X)$ and each component of $\bigcup \mathcal{L}$ intersects both $\{a, b\}$ and $\{c, d\}$ [16, Lemma 6.1.4]. Thus, since X is irreducible between any two points of $\{a, b, c, d\}$, we have that $\bigcup \mathcal{L} = X$. Therefore, $\mathcal{L} \cap F_1(X) \neq \varnothing$ (Theorem 4.19), and $F_2(X) \setminus F_1(X)$ is not continuumwise connected. $\square$

As a consequence of Theorem 4.22, by Lemma 2.4, we obtain [19, Example 4.5].

The next corollary follows from Theorems 4.22 and 4.1.

Corollary 4.23. *Let X be an indecomposable chainable continuum. Then $C_2(X) \setminus C_1(X)$ is not continuumwise connected.*

Since Knaster continua are chainable, we obtain:

Corollary 4.24. *Let X be a Knaster continuum. Then $C_2(X) \setminus C_1(X)$ is not continuumwise connected.*

As a consequence of Theorem 4.1 and Corollary 4.24, we have the next result. This corollary is stronger than Lemma 2.8.

Corollary 4.25. *If X is a Knaster continuum, then $F_2(X) \setminus F_1(X)$ is not continuumwise connected.*

Let X be a continuum and let p be a point of X. Then the *union composant of* p, denoted by $\kappa_u^X(p)$, is:

$$\kappa_u^X(p) = \bigcup\{M \mid M \text{ is a continuum-chainable subcontinuum of } X \text{ and } p \in M\}.$$

We write $\kappa_u(p)$ if there is no possibility of confusion. Note that $\kappa_u(p)$ is a connected subset of X [4, Corollary 6.5].

Notation 4.26. For non-empty closed subsets A and B of a compact metric space (X, d), let $d(A, B) = \inf\{d(a, b) \mid a \in A, b \in B\}$.

Lemma 4.27. *Let X be a continuum with metric d. If $A, B \in C_1(X)$ and $\varepsilon \in (0, d(A, B))$, then $A \cup B \in C_2(X) \setminus \mathcal{N}_{\mathcal{H}}\left(\frac{\varepsilon}{2}; C_1(X)\right)$.*

Proof. Suppose that there exists $E \in C_1(X)$ such that $\mathcal{H}(E, A\cup B) < \frac{\varepsilon}{2}$. Then $E \subseteq \mathcal{N}_d\left(\frac{\varepsilon}{2}; A\right) \cup \mathcal{N}_d\left(\frac{\varepsilon}{2}; B\right)$ and $A \cup B \subseteq \mathcal{N}_d\left(\frac{\varepsilon}{2}; E\right)$. Since the union of the disjoint open subsets $\mathcal{N}_d\left(\frac{\varepsilon}{2}; A\right)$ and $\mathcal{N}_d\left(\frac{\varepsilon}{2}; B\right)$ of X contains the subcontinuum E, either $E \subseteq \mathcal{N}_d\left(\frac{\varepsilon}{2}; A\right)$ or $E \subseteq \mathcal{N}_d\left(\frac{\varepsilon}{2}; B\right)$. Assume $E \subseteq \mathcal{N}_d\left(\frac{\varepsilon}{2}; A\right)$. This inclusion and the fact that $B \subseteq \mathcal{N}_d\left(\frac{\varepsilon}{2}; E\right)$ together imply that $B \subseteq \mathcal{N}_d(\varepsilon; A)$. This contradicts the choice of ε. Therefore, $\mathcal{H}(E, A \cup B) \geq \frac{\varepsilon}{2}$ for each $E \in C_1(X)$. □

Lemma 4.28. *Let X be a continuum. Suppose that there exist a point p in X, a subcontinuum A of X and $\varepsilon > 0$ such that:*

- *$\kappa_u(p)$ is a dense subset of X;*

and

- *$B_{\mathcal{H}}(A; \varepsilon) \cap C_1(\kappa_u(p)) = \varnothing$.*

Then there exists a subcontinuum $\mathcal{L}$ of $C_2(X) \setminus C_1(X)$ such that:

(1) *L has a component in A, for each $L \in \mathcal{L}$;*

and

(2) *$\bigcup \mathcal{L} = X$.*

Proof. Since $\kappa_u(p)$ is a dense subset of X and A is a proper closed subset of X, we may assume that

$$B_d(p; \varepsilon) \cap A = \varnothing. \tag{4.1}$$

We show first that if $x \in \kappa_u(p)$, then there exists a subcontinuum $\mathcal{S}$ of $C_2(X) \setminus \mathcal{N}_{\mathcal{H}}(\frac{\varepsilon}{3}; C_1(X))$ such that $\{p\} \cup A \in \mathcal{S}$, $x \in \bigcup \mathcal{S}$, and each element of $\mathcal{S}$ has a component contained in A.

Since $x \in \kappa_u(p)$, there exists a continuum-chainable subcontinuum N of X such that $x, p \in N$. Let $\{N_1, \ldots, N_m\}$ be an $\frac{\varepsilon}{3}$-chain of continua such that $p \in N_1$, $x \in N_m$. Note that $N_1 \cup \cdots \cup N_m \subseteq N \subseteq \kappa_u(p)$. Thus, each $N_j \notin B_{\mathcal{H}}(A; \varepsilon)$. This implies that either $A \setminus \mathcal{N}_d(\varepsilon; N_j) \neq \varnothing$ or $N_j \setminus \mathcal{N}_d(\varepsilon; A) \neq \varnothing$.

Let $J_1 = \{j \in \{1, \ldots, m\} \mid N_j \subseteq \mathcal{N}_d(\varepsilon; A)\}$ and let $J_2 = \{1, \ldots, m\} \setminus J_1$. We have that $1 \in J_2$ by (4.1). Take $x_1 = p$ and for each $j \in J_2$ with $j \geq 2$, let $x_j \in N_j \setminus \mathcal{N}_d(\varepsilon; A)$ and let $\mathcal{N}_j = \{\{x_j\} \cup E : E \in C_1(A)\}$. We apply Lemma 4.27 to conclude that if $j \in J_2$, then the subcontinuum $\mathcal{N}_j$ of $C_2(X)$ satisfies that $\mathcal{N}_j \cap \mathcal{N}_{\mathcal{H}}\left(\frac{\varepsilon}{2}; C_1(X)\right) = \varnothing$. Let $S_1, S_2, \ldots, S_r$ be the components of $\bigcup_{j \in J_1} N_j$. Observe that each $S_k \subseteq \kappa_u(p)$ and $S_k \subseteq \mathcal{N}_d(\varepsilon; A)$. This implies that there exists $w_k \in A \setminus \mathcal{N}_d(\varepsilon; S_k)$ for each $k \in \{1, 2, \ldots, r\}$. By Lemma 4.27, we obtain that the subcontinuum $\mathcal{S}_k = \{\{w_k\} \cup D : D \in C_1(S_k)\}$ of $C_2(X)$ satisfies that $\mathcal{S}_k \cap \mathcal{N}_{\mathcal{H}}\left(\frac{\varepsilon}{2}; C_1(X)\right) = \varnothing$.

Let $\mathcal{S} = \left(\bigcup_{j\in J_2} \mathcal{N}_j\right) \cup (\bigcup\{\mathcal{S}_k \mid k \in \{1, 2, \ldots, r\}\})$. The subcontinuum $\mathcal{S}$ satisfies the required properties.

Let $\{x_i\}_{i=1}^{\infty}$ be a dense subset of $\kappa_u(p)$. Let $\mathcal{M}_i$ be the continuum constructed in the previous paragraph such that $\{p\} \cup A \in \mathcal{M}_i$, $\mathcal{M}_i \subseteq C_2(X) \setminus \mathcal{N}_{\mathcal{H}}(\frac{\varepsilon}{2}; C_1(X))$ and $x_i \in \bigcup \mathcal{M}_i$, for each $i \in \mathbb{N}$. Let $\mathcal{L} = \mathrm{Cl}_{C_2(X)}(\bigcup_{i=1}^{\infty} \mathcal{M}_i)$. Observe that since $\bigcup_{i=1}^{\infty} \mathcal{M}_i$ is connected, $\mathcal{L}$ is a subcontinuum of $C_2(X) \setminus C_1(X)$. Moreover, $\bigcup_{i=1}^{\infty} \mathcal{M}_i \subseteq C_2(X) \setminus \mathcal{N}_{\mathcal{H}}(\frac{\varepsilon}{2}; C_1(X))$ and $C_2(X) \setminus \mathcal{N}_{\mathcal{H}}(\frac{\varepsilon}{2}; C_1(X))$ is closed. Thus, $\mathcal{L} \subseteq C_2(X) \setminus \mathcal{N}_{\mathcal{H}}(\frac{\varepsilon}{2}; C_1(X))$. Let $M \in \mathcal{M}_i$, for some $i \in \mathbb{N}$. Let $M = R \cup S$, where R and S are the components of M. We know that either $R \subseteq A$ or $S \subseteq A$. Thus, if $L \in \mathcal{L}$, then L has a component in A. We have condition (1). Also, $\bigcup \mathcal{L}$ is a compact subset of X containing $\{x_i\}_{i=1}^{\infty}$. Therefore, $X = \bigcup \mathcal{L}$. □

Lemma 4.29. *Let X be a continuum. Suppose that there exist a point p in X, a subcontinuum A of X and $\varepsilon > 0$ such that:*

- *$\kappa_u(p)$ is a dense subset of X;*

and

- *$B_{\mathcal{H}}(A; \varepsilon) \cap C_1(\kappa_u(p)) = \varnothing$.*

If $C_1, \ldots, C_n$ are subcontinua of $X \setminus A$, then there exists a subcontinuum $\mathcal{L}$ of $C_2(X) \setminus C_1(X)$ such that $\bigcup \mathcal{L} = X$ and $\{A \cup C_1, \ldots, A \cup C_n\} \subseteq \mathcal{L}$.

Proof. By Lemma 4.28, there exists a subcontinuum $\mathcal{S}$ of $C_2(X) \setminus C_1(X)$ such that S has a component in A, for each $S \in \mathcal{S}$, and $\bigcup \mathcal{S} = X$. Let $S = S_1 \cup S_2 \in \mathcal{S}$ be such that $S_1 \subseteq A$ and $S_2 \cap C_1 \neq \varnothing$. Let $a \in S_1 \subseteq A$ and $z \in S_2 \cap C_1$. By [22, Theorem (1.8)], there exist order arcs α_1, α_2, β_1 and β_2 from $\{a\}$ to A, from $\{a\}$ to S_1, from $\{z\}$ to C_1 and from $\{z\}$ to S_2, respectively. Let

$$\mathcal{J}_1 = \{M \cup N \mid M \in \alpha_1 \text{ and } N \in \beta_1\} \cup \{M \cup N \mid M \in \alpha_2 \text{ and } N \in \beta_2\}.$$

Observe that $\mathcal{J}_1$ is a subcontinuum of $C_2(X) \setminus C_1(X)$ such that $A \cup C_1 \in \mathcal{J}_1$ and $S \in \mathcal{J}_1 \cap \mathcal{S}$. Hence, $\mathcal{S} \cup \mathcal{J}_1$ is a subcontinuum of $C_2(X) \setminus C_1(X)$.

Similarly, for each $i \in \{2, \ldots, n\}$, there exists a subcontinuum $\mathcal{J}_i$ of $C_2(X) \setminus C_1(X)$ such that $A \cup C_i \in \mathcal{J}_i$ and $\mathcal{J}_i \cap \mathcal{S} \neq \varnothing$. Therefore, $\mathcal{L} = \mathcal{S} \cup \mathcal{J}_1 \cup \cdots \cup \mathcal{J}_n$ is a subcontinuum of $C_2(X) \setminus C_1(X)$ such that $\bigcup \mathcal{L} = X$ and $\{A \cup C_1, \ldots, A \cup C_n\} \subseteq \mathcal{L}$. □

Theorem 4.30. *Let X be a continuum such that $X = I \cup R$, where $R \cong [0, \infty)$, $I = \mathrm{Cl}_X(R) \setminus R$ and I is arcwise connected. If $C_1(R)$ is not dense in $C_1(X)$, then $C_2(X) \setminus C_1(X)$ is continuumwise connected.*

Proof. Since $C_1(R)$ is not dense in $C_1(X)$, we have that there exist a proper subcontinuum A of I and $\varepsilon > 0$ such that $B_{\mathcal{H}}(A; \varepsilon) \cap C_1(R) = \varnothing$. By Lemma 4.28, there exists a subcontinuum $\mathcal{L}$ of $C_2(X) \setminus C_1(X)$ such that L has a component in A, for each $L \in \mathcal{L}$, and $\bigcup \mathcal{L} = X$. Let $B \in C_2(X) \setminus C_1(X)$ and let C and D be the components of B. We show that there exists a subcontinuum $\mathcal{S}$ of $C_2(X) \setminus C_1(X)$ such that $\mathcal{S} \cap \mathcal{L} \neq \varnothing$ and $B \in \mathcal{S}$. We consider two cases:

Case (1). $B \subseteq I$.

Since I is arcwise connected, $C_2(I) \setminus C_1(I)$ is continuumwise connected. Note that $\mathcal{L} \cap (C_2(I) \setminus C_1(I)) \neq \varnothing$. Hence, there exists a subcontinuum $\mathcal{S}$ of $C_2(I) \setminus C_1(I)$ such that $B \in \mathcal{S}$ and $\mathcal{S} \cap \mathcal{L} \neq \varnothing$.

Case (2). $B \cap R \neq \varnothing$.

Note that there exists a subcontinuum M of X such that either $C \cup A \subseteq M$ and $D \cap M = \varnothing$, or $D \cup A \subseteq M$ and $C \cap M = \varnothing$. Suppose that $C \cup A \subseteq M$ and $D \cap M = \varnothing$. Let

$$\mathcal{S} = \{D \cup R \mid R \in C_1(M)\}.$$

Observe that $\{B, D \cup A\} \subseteq \mathcal{S}$. Also, by Lemma 4.29, we may assume that $D \cup A \in \mathcal{L}$. Thus, $\mathcal{S}$ is a subcontinuum of $C_2(X) \setminus C_1(X)$ such that $\mathcal{S} \cap \mathcal{L} \neq \varnothing$ and $B \in \mathcal{S}$. □

In Corollary 2.9 it is shown that if X is the $\sin(1/x)$-continuum, then $F_2(X) \setminus F_1(X)$ is not continuumwise connected. Next theorem follows from Theorems 4.1 and 4.30. It shows that for some compactifications of $[0, \infty)$, $F_2(X) \setminus F_1(X)$ is continuumwise connected.

Theorem 4.31. *Let X be a continuum such that $X = I \cup R$, where $R \cong [0, \infty)$, $I = \mathrm{Cl}_X(R) \setminus R$ and I is arcwise connected. If $C_1(R)$ is not a dense subset of $C_1(X)$, then $F_2(X) \setminus F_1(X)$ is continuumwise connected.*

Note that if X is a ray converging to a simple triod (see the continuum pictured in Figure 1 below), X contains a terminal subcontinuum of X and, by Theorem 4.31, $F_2(X) \setminus F_1(X)$ is continuumwise connected.

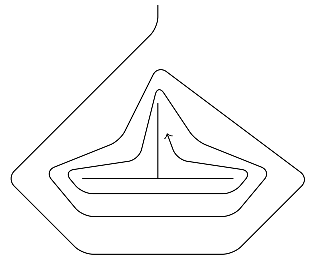

Figure 1.

Theorem 4.32. *Let X be an indecomposable continuum such that each composant of X is a union composant. Then one of the following holds:*

(1) $\langle \kappa_1, \kappa_2 \rangle \cap C_2(X)$ *is dense in* $C_2(X)$, *for each two different composants* κ_1 *and* κ_2 *of* X,

or

(2) *There exists a subcontinuum* $\mathcal{L}$ *of* $C_2(X) \setminus C_1(X)$ *such that* $\bigcup \mathcal{L} = X$.

Proof. Suppose that there exist two different composants κ_1 and κ_2 of X such that $\langle \kappa_1, \kappa_2 \rangle \cap C_2(X)$ is not dense in $C_2(X)$; i.e., there exist $A \in C_2(X) \setminus C_1(X)$ and $\varepsilon > 0$ such that $B_{\mathcal{H}}(A; \varepsilon) \cap \langle \kappa_1, \kappa_2 \rangle = \varnothing$. Let A_1 and A_2 be the components of A. Without loss of generality, we may assume that $B_{\mathcal{H}}(A_1; \varepsilon) \cap C_1(\kappa_1) = \varnothing$. Since the composants of X are dense in X [12, Theorem 3-44], now the theorem follows from Lemma 4.28. □

Remark 4.33. By Theorem 4.32, there exists a subcontinuum $\mathcal{L}$ of $C_2(X) \setminus C_1(X)$, where X is the continuum in Figure 2, such that $\bigcup \mathcal{L} = X$. (Compare with Theorem 4.19.)

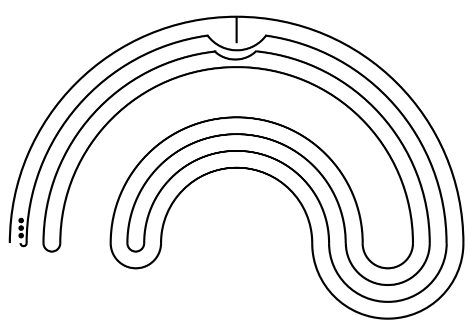

Figure 2.

5. $C_1(X)$ is a nonblock continuum

The following result is [19, Theorem 3.5].

Theorem 5.1. *Let X be a continuum. Then $F_1(X)$ is a nonblock subcontinuum of $F_2(X)$.*

We have the following natural question.

Question 5.2. *Given a continuum X, is $C_1(X)$ a nonblock subcontinuum of $C_2(X)$?*

In this section, we give partial answers to Question 5.2.

A continuum X has the *property of Kelley at a point* $p \in X$ provided that for every $\varepsilon > 0$, there exists a $\delta > 0$ such that if A is a subcontinuum of X and $p \in A$, then for each $x \in X$ such that $d(x, p) < \delta$, there exists a subcontinuum B_x of X such that $x \in B_x$ and $\mathcal{H}(A, B_x) < \varepsilon$ [23, p. 292]. The number δ in the definition is called a *Kelley number* for (X, ε, p). A continuum X has the *property of Kelley* provided that X has the property of Kelley at each of its points; in this case, for each $\varepsilon > 0$, it follows from compactness (using the Lebesgue number of a cover [16, Theorem 1.6.6]) that there is a $\delta > 0$ that is a Kelley number for (X, ε, p), for all $p \in X$, which we call a *Kelley number* for (X, ε).

A continuum X has the *property of Kelley weakly* provided that there exists a dense subset $\mathcal{A}$ of $C_1(X)$ such that X has the property of Kelley at some point of each $A \in \mathcal{A}$.

A proof of the following result can be found in [17, Corollary 4.11]. We include its proof for the convenience of the reader.

Lemma 5.3. *Let X be a continuum with the property of Kelley weakly and let n be a positive integer. Then for each open subset $\mathcal{U}$ of $C_n(X)$, there exists $A \in \mathcal{U}$ such that X has the property of Kelley at some point a of each component of A.*

Proof. Let $\mathcal{U}$ be an open subset of $C_n(X)$, let $K \in \mathcal{U}$ and let $V_1, \ldots, V_t$ be open subsets of X such that $K \in \langle V_1, \ldots, V_t \rangle \cap C_n(X) \subseteq \mathcal{U}$. Let L be a component of K and let $V_{j_1}, \ldots, V_{j_l}$ denote all the elements of $V_1, \ldots, V_t$ that L intersects. Then $L \in \langle V_{j_1}, \ldots, V_{j_l} \rangle \cap C_1(X)$. Since X has the property of Kelley weakly, there exists $M \in C_1(X)$ such that $M \in \langle V_{j_1}, \ldots, V_{j_l} \rangle \cap C_1(X)$ and X has the property of Kelley at some point a of M. Let $A_1 = (K \setminus L) \cup M$. Then $A_1 \in \langle V_1, \ldots, V_t \rangle \cap C_n(X) \subseteq \mathcal{U}$ and X has the property of Kelley at $a \in M \subseteq A_1$. Repeat this process with each component L' of K to obtain an element R of $\mathcal{U}$ satisfying that X has the property of Kelley at some point of each component of R. □

Theorem 5.4. *If X is a continuum with the property of Kelley weakly, then $C_1(X)$ is a nonblock subcontinuum of $C_2(X)$.*

Proof. Since $F_1(X)$ is a nonblock subcontinuum of $F_2(X)$ [19, Theorem 3.5], there exists an increasing sequence of subcontinua $\{\mathcal{L}_k\}_{k=1}^{\infty}$ of $F_2(X) \setminus F_1(X)$ such that $\bigcup_{k=1}^{\infty} \mathcal{L}_k$ is a dense subset of $F_2(X) \setminus F_1(X)$. Let $\{\mathcal{U}_k\}_{k=1}^{\infty}$ be a decreasing sequence of open subsets of $C_2(X)$ such that $C_1(X) = \bigcap_{k=1}^{\infty} \mathcal{U}_k$ and $\mathrm{Cl}_{C_2(X)}(\mathcal{U}_k) \cap \mathcal{L}_k = \varnothing$, for each $k \in \mathbb{N}$. Given $k \in \mathbb{N}$, let $\mathcal{M}_k$ be the component of the closed subset $C_2(X) \setminus \mathcal{U}_k$ that contains $\mathcal{L}_k$. Hence, $\{\mathcal{M}_k\}_{k=1}^{\infty}$ is an increasing sequence of subcontinua of $C_2(X) \setminus C_1(X)$.

To see that $\bigcup_{k=1}^{\infty} \mathcal{M}_k$ is a dense subset of $C_2(X) \setminus C_1(X)$, let $A \in C_2(X) \setminus C_1(X)$, whose components are A_1 and A_2, satisfying that X has the property of Kelley at $a_j \in A_j$, $j \in \{1, 2\}$, (Lemma 5.3) and let $\varepsilon > 0$. Since $C_1(X)$ is closed, we may assume that $\mathcal{B}_{\mathcal{H}}(\varepsilon; A) \cap C_1(X) = \varnothing$. Let $\delta > 0$ be a Kelley number for a_1, a_2 and ε. Note that $\{a_1, a_2\}$ is an element of $F_2(X) \setminus F_1(X)$. Since $\mathrm{Cl}_{F_2(X)}(\bigcup_{k=1}^{\infty} \mathcal{L}_k) = F_2(X)$, we obtain that there exists $\{b_1, b_2\} \in \mathcal{L}_m$, for some $m \in \mathbb{N}$, such that $\mathcal{H}(\{a_1, a_2\}, \{b_1, b_2\}) < \delta$. Thus, there exist $B_1, B_2 \in C_1(X)$ such that $b_1 \in B_1$, $b_2 \in B_2$, $\mathcal{H}(A_1, B_1) < \varepsilon$ and $\mathcal{H}(A_2, B_2) < \varepsilon$. Let $B = B_1 \cup B_2$. Hence, $\mathcal{H}(A, B) < \varepsilon$, and B belongs to $C_2(X) \setminus C_1(X)$. Let α be an order arc from $\{b_1, b_2\}$ to B [22, Theorem (1.8)]. Then each element of α is a closed subset of X having exactly two components. Thus, from the fact that $\{C_2(X) \setminus \mathrm{Cl}_{C_2(X)}(\mathcal{U}_k)\}_{k=1}^{\infty}$ is an increasing sequence consisting of open subsets covering α, we have that there exists $j \in \mathbb{N}$ such that $j \geq m$ and $\alpha \subseteq C_2(X) \setminus \mathrm{Cl}_{C_2(X)}(\mathcal{U}_j)$. Since $\alpha \cap \mathcal{L}_j \neq \varnothing$, $\alpha \subseteq \mathcal{M}_j$. Hence, $\mathcal{B}_{\mathcal{H}}(\varepsilon; A) \cap \mathcal{M}_j \neq \varnothing$, and $\bigcup_{k=1}^{\infty} \mathcal{M}_k$ is a dense subset of $C_2(X) \setminus C_1(X)$. Therefore, $C_1(X)$ is a nonblock subcontinuum of $C_2(X)$. □

Corollary 5.5. *If X is a continuum with the property of Kelley, then $C_1(X)$ is a nonblock subcontinuum of $C_2(X)$.*

By Lemma 2.4, if $C_2(X) \setminus C_1(X)$ is continuumwise connected, then $C_1(X)$ is a nonblock subcontinuum of $C_2(X)$. In the next example, we show a continuum with the property of Kelley such that $C_2(X) \setminus C_1(X)$ is not continuumwise connected. Thus, Theorem 5.4 cannot be improved.

Example 5.6. Let $X = \{(x, \sin(\frac{1}{x})) \mid x \in (0, 1]\} \cup (\{0\} \times [-1, 1])$. Then X is a continuum with the property of Kelley. By Corollary 2.9, $F_2(X) \setminus F_1(X)$ is not continuumwise connected. Note that, by Theorem 4.1, we obtain that $C_2(X) \setminus C_1(X)$ is not continuumwise connected.

Theorem 5.7. *Let X be a continuum such that $X = I \cup R$, where $R \cong [0, \infty)$, $I = \mathrm{Cl}_X(R) \setminus R$ and I is arcwise connected. If $C_1(R)$ is dense in $C_1(X)$, then $C_1(X)$ is a nonblock subcontinuum of $C_2(X)$.*

Proof. Suppose that $C_1(R)$ is dense in $C_1(X)$. Let $K \in C_2(R) \setminus C_1(R)$. We show that $\kappa_{C_2(X)\setminus C_1(X)}(K)$ is a dense subset of $C_2(X)$ (Lemma 2.3). Let $B = C \cup D$ be an element of $C_2(X) \setminus C_1(X)$ and let $\varepsilon > 0$. Since $C_1(R)$ is dense in $C_1(X)$, there exist C' and D' in $C_1(R)$ such that $\mathcal{H}(C, C') < \varepsilon, \mathcal{H}(D, D') < \varepsilon$ and $C' \cap D' = \varnothing$. Since $R \cong [0, \infty)$, there exists an arc $J \subseteq R$ such that $K \cup C' \cup D' \subseteq J$. Note that, by Corollary 3.7, $C_2(J) \setminus C_1(J)$ is continuumwise connected (an arc is an aposyndetic continuum). Hence, there exists a subcontinuum $\mathcal{L}$ of $C_2(J) \setminus C_1(J)$, where $\{K, C' \cup D'\} \subseteq \mathcal{L}$. Note that $C' \cup D' \in \mathcal{L} \cap B_{\mathcal{H}}(B; \varepsilon)$. Therefore, $\kappa_{C_2(X)\setminus C_1(X)}(K)$ is a dense subset of $C_2(X)$ and $C_1(X)$ is a nonblock subcontinuum in $C_2(X)$ (Lemma 2.3). □

Lemma 5.8. *Let X be a chainable indecomposable continuum, let A be a proper subcontinuum of X and let κ be a composant of X. Then there exists a sequence $\{K_n\}_{n=1}^{\infty}$ of subcontinua of κ converging to A.*

Proof. We may assume that $A \cap \kappa = \varnothing$. Let $\varepsilon > 0$. Since X is chainable, there exist open connected subsets $U_1, \ldots, U_n$ of $\mathbb{R}^2$ such that $\operatorname{diam}(U_i) < \frac{\varepsilon}{2}$, for each $i \in \{1, \ldots, n\}$, $X \subseteq U_1 \cup \cdots \cup U_n$, and $\{U_1, \ldots, U_n\}$ is a chain [2, Theorem 5]. Let $\{i_1, \ldots, i_l\} \subseteq \{1, \ldots, n\}$ be such that $A \in \langle U_{i_1}, \ldots, U_{i_l} \rangle$. Note that $U_{i_1} \cup \cdots \cup U_{i_l}$ is connected. Since κ is dense, there exists a subcontinuum L of κ such that $L \cap U_{i_j} \neq \varnothing$, for every $j \in \{1, \ldots, l\}$. By the Boundary Bumping Theorem [18, Theorem 1.4.36], and the connectedness of L, we have that there exists a component K of $L \setminus (U_{i_1} \cup U_{i_l})$ such that $K \cap \operatorname{Cl}_{\mathbb{R}^2}(U_{i_1}) \neq \varnothing$ and $K \cap \operatorname{Cl}_{\mathbb{R}^2}(U_{i_l}) \neq \varnothing$. Since $\operatorname{diam}(U_i) < \frac{\varepsilon}{2}$, for all $i \in \{1, \ldots, n\}$, $\mathcal{H}(A, K) < \varepsilon$. Also, ε was arbitrary and $K \subseteq \kappa$. Thus, there exists a sequence $\{K_n\}_{n=1}^{\infty}$ of subcontinua of κ converging to A. □

In the following theorems, we show sufficient conditions to have that $C_1(X)$ is a nonblock subcontinuum of $C_2(X)$.

Theorem 5.9. *If X is a chainable indecomposable continuum, then $C_1(X)$ is a nonblock subcontinuum of $C_2(X)$.*

Proof. Let κ_1 and κ_2 be two different composants of X. We show that $\langle \kappa_1, \kappa_2 \rangle \cap C_2(X)$ is dense in $C_2(X) \setminus C_1(X)$. Let $A = A_1 \cup A_2$ be a point in $C_2(X) \setminus C_1(X)$ and let $\varepsilon > 0$. By Lemma 5.8, there exist two subcontinua K_1 of κ_1 and K_2 of κ_2, such that $\mathcal{H}(A_1, K_1) < \varepsilon$ and $\mathcal{H}(A_2, K_2) < \varepsilon$. Hence, $\mathcal{H}(A, K_1 \cup K_2) < \varepsilon$. Note that $K_1 \cup K_2 \in \langle \kappa_1, \kappa_2 \rangle \cap C_2(X)$. Thus, $\langle \kappa_1, \kappa_2 \rangle \cap C_2(X)$ is dense in $C_2(X) \setminus C_1(X)$. □

The following result is [2, Theorem 4].

Theorem 5.10. *Suppose X is a circularly chainable continuum and let $\{D_m\}_{m=1}^{\infty}$ be a sequence of circular chains covering X such that* $\text{mesh}(D_m)$ *approaches* 0 *as m increases without limit, and D_{m+1} circles D_m exactly once. Then there exists an embedding h of X into the plane such that for each positive number ε, $h(X)$ can be covered by a circular chain each of whose links is the interior of a round disk of diameter less than ε.*

Using Theorem 5.10, the proof of the following lemma is similar to the one given for Lemma 5.8.

Lemma 5.11. *Let X be a circularly chainable indecomposable continuum as in the statement of Theorem 5.10, let A be a proper subcontinuum of X and let κ be a composant of X. Then there exists a sequence $\{K_n\}_{n=1}^{\infty}$ of subcontinua of κ converging to A.*

The proof of the following result is similar to the one given for Theorem 5.9.

Theorem 5.12. *Let X be an indecomposable circularly chainable continuum as in the statement of Theorem 5.10. Then $C_1(X)$ is a nonblock subcontinuum of $C_2(X)$.*

Theorem 5.13. *Let X be an indecomposable continuum such that each composant of X is a union composant. Then $C_1(X)$ is a nonblock subcontinuum of $C_2(X)$.*

Proof. Let $p \in X$. We consider two cases:

Case (1). $C_1(\kappa_u(p))$ is a dense subset of $C_1(X)$.

Let $B \in C_2(\kappa_u(p)) \setminus C_1(\kappa_u(p))$. We see that $\kappa_{C_2(X)\setminus C_1(X)}(B)$ is dense in $C_2(X) \setminus C_1(X)$. Let $D \in C_2(X) \setminus C_1(X)$ and let $\varepsilon > 0$. Since $C_1(\kappa_u(p))$ is dense, there exists $D' \in C_2(\kappa_u(p)) \setminus C_1(\kappa_u(p))$ such that $\mathcal{H}(D, D') < \varepsilon$. Let B_0 and D'_0 be in $F_2(X) \setminus F_1(X)$ such that $B_0 \subseteq B$, $D'_0 \subseteq D'$ and there exist order arcs α from B_0 to B, and γ from D'_0 to D' [22, Theorem (1.8)]. Let M be a continuum-chainable subcontinuum of $\kappa_u(p)$ such that $B_0 \cup D'_0 \subseteq M$ [4, Lemma 3.2]. By Corollary 4.8, $C_2(M) \setminus C_1(M)$ is continuumwise connected. Thus, there exists a subcontinuum $\mathcal{R}$ of $C_2(M) \setminus C_1(M)$ such that $B_0, D'_0 \in \mathcal{R}$. Note that if $\mathcal{K} = \mathcal{R} \cup \alpha \cup \gamma$, then $\mathcal{K} \subseteq \kappa_{C_2(X)\setminus C_1(X)}(B)$ and $\mathcal{K} \cap B_{\mathcal{H}}(D; \varepsilon) \neq \varnothing$. Therefore, $C_1(X)$ is a nonblock subcontinuum in $C_2(X)$ (Lemma 2.3).

Case (2). $C_1(\kappa_u(p))$ is not dense.

Let $A \in C_1(X)$ and let $\varepsilon > 0$ be such that $B_{\mathcal{H}}(A; \varepsilon) \cap C_1(\kappa_u(p)) = \varnothing$. By Lemma 4.28, there exists a subcontinuum $\mathcal{L}_A$ of $C_2(X) \setminus C_1(X)$ such that $\bigcup \mathcal{L}_A = X$ and if $L = L_1 \cup L_2 \in \mathcal{L}_A$, then $L_1 \subseteq A$.

We prove that $\kappa_{C_2(X)\setminus C_1(X)}(L)$ is dense, where L is any element of $\mathcal{L}_A$. Let $B \in C_2(X)\setminus C_1(X)$ and let $\delta > 0$. In order to prove that $\kappa_{C_2(X)\setminus C_1(X)}(L)$ is dense, we show that there exists a subcontinuum $\mathcal{S}$ of $C_2(X) \setminus C_1(X)$ such that $\mathcal{S} \cap \mathcal{L}_A \neq \varnothing$ and $\mathcal{S} \cap B_{\mathcal{H}}(B;\delta) \neq \varnothing$. Let σ be the composant of X such that $A \subseteq \sigma$. We consider two subcases:

Subcase (2.1). $B \in \mathrm{Cl}_{C_2(X)}(C_2(\sigma))$.

Let $D \in C_2(\sigma) \setminus C_1(\sigma)$ be such that $\mathcal{H}(B, D) < \varepsilon$, and let $L' \in \mathcal{L}_A$, where $L' \subseteq \sigma$. Let M be a continuum-chainable subcontinuum of $\kappa_u(p)$ such that $D \cup L' \subseteq M$. By Corollary 4.8, $C_2(M) \setminus C_1(M)$ is continuumwise connected. Thus, there exists a subcontinuum $\mathcal{S}$ of $C_2(M) \setminus C_1(M)$ such that $D, L' \in \mathcal{S}$. Thus, $\mathcal{S}$ is a subcontinuum of $C_2(X) \setminus C_1(X)$ where $\mathcal{L}_A \cap \mathcal{S} \neq \varnothing$ and $\mathcal{S} \cap B_{\mathcal{H}}(B;\delta) \neq \varnothing$.

Subcase (2.2). $B = C \cup D$ and $D \notin \mathrm{Cl}_{C_2(X)}(C_1(\sigma))$.

Let $\mathcal{L}_D$ be a subcontinuum of $C_2(X) \setminus C_1(X)$ such that $\bigcup \mathcal{L}_D = X$ (see Lemma 4.28). By Lemma 4.29, we may assume that $D \cup A \in \mathcal{L}_A \cap \mathcal{L}_D$ and $B \in \mathcal{L}_D$. Thus, $\mathcal{S} = \mathcal{L}_D$ is a subcontinuum of $C_2(X) \setminus C_1(X)$ such that $\mathcal{S} \cap \mathcal{L}_A \neq \varnothing$ and $\mathcal{S} \cap B_{\mathcal{H}}(B;\delta) \neq \varnothing$.

Therefore, $\kappa_{C_2(X)\setminus C_1(X)}(L)$ is a dense subset of $C_2(X)$ and $C_1(X)$ is a nonblock subcontinuum of $C_2(X)$ (Lemma 2.3). □

Question 5.14. *Let X be a decomposable continuum. Then does it follow that $C_1(X)$ is a nonblock continuum in $C_2(X)$?*

Theorem 5.15. *Let X be a decomposable continuum. The following are equivalent:*

(1) *$C_1(X)$ is a nonblock subcontinuum of $C_2(X)$;*

(2) *$C_1(X)$ is a shore subcontinuum of $C_2(X)$;*

(3) *$C_1(X)$ is not a strong center of $C_2(X)$.*

Proof. By Lemma 2.4, we only need to show that (3) implies (1). Let A and B be proper subcontinua of X such that $X = A \cup B$. Let $a \in A \setminus B$ and $b \in B \setminus A$. Let U and V be open subsets of X such that $a \in U \subseteq \mathrm{Cl}_X(U) \subseteq X \setminus B$ and $b \in V \subseteq \mathrm{Cl}_X(V) \subseteq X \setminus A$. We define:

$$\mathcal{L} = \langle \mathrm{Cl}_X(U), B\rangle \cup \langle A, \mathrm{Cl}_X(V)\rangle.$$

Note that $\mathcal{L}$ is a compact subset of $C_2(X) \setminus C_1(X)$ and $\langle U, V\rangle \subseteq \mathcal{L}$. Using order arcs is not difficult to prove that $\mathcal{L}$ is arcwise connected.

We see that $\kappa_{C_2(X)\setminus C_1(X)}(L)$ is a dense subset of $C_2(X) \setminus C_1(X)$, for each $L \in \mathcal{L}$. Let $L \in \mathcal{L}$ and let $\mathcal{U}$ be an open subset of $C_2(X)\setminus C_1(X)$. Since $C_1(X)$ is not a strong center of $C_2(X)$, there exists a subcontinuum $\mathcal{S}$ of $C_2(X)\setminus C_1(X)$ such that $\mathcal{S}\cap\mathcal{U}\neq\varnothing$ and $\mathcal{S}\cap\mathrm{Int}_{C_2(X)}(\mathcal{L})\neq\varnothing$. Hence, $\mathcal{L}\cup\mathcal{S}$ is a subcontinuum of $C_2(X)\setminus C_1(X)$ containing L. Therefore, $\mathcal{L}\cup\mathcal{S}\subseteq\kappa_{C_2(X)\setminus C_1(X)}(L)$ and $\kappa_{C_2(X)\setminus C_1(X)}(L)$ is a dense subset of $C_2(X)\setminus C_1(X)$. We have that $C_1(X)$ is a nonblock subcontinuum of $C_2(X)$ (Lemma 2.3). □

Acknowledgements. *The authors thank the referee for the valuable suggestions made that improved the paper. This work was supported by UNAM PASPA-DGAPA. The first named author thanks the financial support provided by Proyecto 4247 VIE-UIS de La Vicerrectoría de Investigación y Extensión de la Universidad Industrial de Santander. The second named author thanks the Universidad Industrial de Santander, Colombia, for the support given during this research.*

References

[1] J. G. Anaya and D. Maya, Non-cut ordered arcs of the hyperspace of subcontinua, Topology Appl. 349 (2024), 108908.

[2] R. H. Bing, Embedding circle-like continua in the plane, Canadian J. Math. 14 (1962), 113–128.

[3] J. Bobok, P. Pyrih, and B. Vejnar, Non-cut shore and non-block points in continua, Glasnik Mat. Ser. III 51 (2016), 237–253.

[4] J. Camargo, R. A. López and S. Macías, On weakly continuum-chainable and almost continuum-chainable continua, Topology Appl. 345 (2024), 108833.

[5] J. Camargo and S. Macías, Quotients of n-fold hyperspaces, Topology Appl. 197 (2016), 154–166.

[6] J. Camargo and S. Macías, On Wilder, Strongly Wilder, Continuumwise Wilder, D, D^*, and D^{**} continua, Bull. Malays. Math. Sci. Soc. 47, (2024), 95.

[7] J. Camargo and S. Macías, More on quotients of n-fold hyperspaces, Houston J. Math. 50 (2024), 443–475.

[8] J. Camargo and S. Macías, More on $\mathcal{T}$-closed sets, preprint.

[9] J. Camargo, D. Maya and P. Pellicer-Covarrubias, Noncut subsets of the hyperspace of subcontinua, Topology Appl. 305 (2022), 107867.

[10] B. Espinoza and E. Matsuhashi, D-continua, $D^\star$-continua, and Wilder continua, Topology Appl. 285 (2020), 107393.

[11] G. Higuera and A. Illanes, Induced mappings on symmetric products, Topology Proc. 37 (2011), 367–401.

[12] J. G. Hocking and G. S. Young, Topology, Dover Publications, Inc., New York, 1988.

[13] A. Illanes, V. Martínez-de-la-Vega, J. M. Martínez-Montejano and D. Michalik, Diagonals separating the square of a continuum, Bull. Malays. Math. Sci. Soc. 46 (2023), 170.

[14] A. Illanes, V. Martínez-de-la-Vega, J. M. Martínez-Montejano and D. Michalik, Diagonals of the square of a continuum, Topology Appl. 364 (2025), 109099.

[15] S. Macías, Aposyndetic properties of symmetric products of continua, Topology Proc. 22 (1997), 281–296.

[16] S. Macías, Topics on Continua, 2nd Edition, Springer-Cham, (2018).

[17] S. Macías, The property of Kelley and continua, Rev. Integr. Temas Mat. 37, no. 1 (2019), 17-29.

[18] S. Macías, Set Function $\mathcal{T}$: An Account on F. B. Jones' Contributions to Topology, Series Developments in Mathematics, vol. 67, Springer, (2021).

[19] V. Martínez-De-La-Vega and J. Martínez-Montejano, Concerning when $F_1(X)$ is a continuum of colocal connectedness in hyperspaces and symmetric products, Colloquium Math. 160 (2020), 297–307.

[20] J. Martínez-Montejano, Mutual aposyndesis of symmetric products, Topology Proc. 24 (1999), 203–213.

[21] R. L. Moore, Foundations of point set theory, Colloquim Publications, vol. 13, American Mathematical Society, Providence, Rhode Island (1962).

[22] S. B. Nadler, Jr., Hyperspaces of Sets: A Text with Research Questions, Monographs and Textbooks in Pure and Applied Mathematics, vol. 49, Marcel Dekker, NewYork, Basel, (1978). Reprinted in: Aportaciones Matemáticas de la Sociedad Matemática Mexicana, Serie Textos #33, (2006).

[23] R. W. Wardle, On a property of J. L. Kelley, Houston J. Math. 3 (1977), 291–299.

[24] G. T. Whyburn, Analytic topology, Colloquim Publications, vol. 28, American Mathematical Society, Providence, Rhode Island (1942).